Exercise 25

Give an alternative solution to Example 3 by letting $y = \sinh^{-1} x$ and then using Exercise 9 and Example 1(a) with x replaced by y.

Solution

The aim in Example 3 is to show that

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Let

$$y = \sinh^{-1} x.$$

Then

$$\sinh y = x.$$

Use the result from Example 1(a), $\cosh^2 y - \sinh^2 y = 1$, to get a formula for $\cosh y$.

$$\cosh y = \pm\sqrt{\sinh^2 y + 1}$$

Hyperbolic cosine is always positive, so discard the minus sign.

$$\cosh y = \sqrt{\sinh^2 y + 1}$$

$$= \sqrt{x^2 + 1}$$

According to Exercise 9, $\cosh y + \sinh y = e^y$, so

$$e^y = \sinh y + \cosh y$$

$$= x + \sqrt{x^2 + 1}.$$

Take the natural logarithm of both sides to solve for y.

$$\ln e^y = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$y \ln e = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$y = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Therefore,

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$